Complex Modal Estimation of Wave Parameters in One-Dimensional Media

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A method of complex orthogonal decomposition (COD) [1] is applied to the extraction of modes from simulation data of multi-modal traveling waves in one-dimensional continua.

In applying COD, the first step is to express a real oscillatory signals $y_j = y(x_j, t)$, $j = 1, \ldots, M$, where M is the number of sensors distributed on the structure or specemin, as complex analytic signals $z_j(t)$, which are sampled N times to generate vectors $\mathbf{z}_j = [z_j(t_1) \cdots z_j(t_N)]^T$. We build an $M \times N$ complex ensemble matrix $Z = [\mathbf{z}_1 \cdots \mathbf{z}_M]^T$. We then construct a complex Hermitian correlation matrix $\mathbf{R} = \frac{1}{N}\mathbf{Z}\bar{\mathbf{Z}}^T$, where the bar indicates complex conjugation. We find real eigenvalues λ_j and complex eigenvectors $\mathbf{u}_j$ of $\mathbf{R}$. The eigenvalues and eigenvectors are referred to as complex orthogonal values (COVs) and modes (COMs), respectively. The COVs, $\lambda_j = d_j M/L$, are proportional to the mean squared modal amplitudes d_j, where L is the length of the domain for one-dimensional media.

We can write the complex ensemble as $\mathbf{Z} = \mathbf{UQ}$, where $\mathbf{U}$ has columns of COD orthogonal modes, and $\mathbf{Q}$ is the ensemble of complex orthogonal modal coordinates. If the modes in $\mathbf{U}$ are normalized, then by complex orthogonality,

$$\mathbf{Q} = \bar{\mathbf{U}}^T\mathbf{Z}$$

is the complex modal coordinate ensemble matrix, the rows of which are the samples of each modal coordinate, $q_j(t)$, sampled at $t = t_1, \ldots, t_N$.

From the modal coordinates in ensemble $\mathbf{Q}$, frequency information can be obtained (e.g. by FFT or complex whirl rate for nearly harmonic signals) for the wave components. Likewise, the wave number γ_j (2π over the wavelength) can be obtained from each of the complex modes $\mathbf{u}_j$. The wave speed (phase velocity) of each wave component is then $c_j = \omega_j/\gamma_j$.

1. Numerical Example: Two-Harmonic Wave

Complex wave modes were extracted from a two-wave simulation of a dispersive medium, whose frequency, wavenumber, and phase velocity relationship matched the theory of an infinite uniform Euler-Bernoulli beam. The wave had the form $y(x,t) = A_1 \sin(\gamma_1 x - \omega_1 t) + A_2 \sin(\gamma_2 x - \omega_2 t)$, where $A_1 = 1, A_2 = 1/2, \gamma_1 = 20$ rad/m, and $\gamma_2 = 16$ rad/m. Consistent with Euler-Bernoulli theory for a steel beam with a Young's modulus $E = 200e9$ N/m^2, rectangular-cross-sectional area moment of inertia $I = bh^3/12$ with width $b = 1$ cm and height $h = 1$ mm, and mass density $\rho = 7860$ kg/m^3, the wave frequency is related to wave number as $\omega = a\gamma^2$, where $a = \sqrt{EI/\rho A} = 1.4562$ m/s. Therefore, $\omega_1 = 582.4694$ rad/s and $\omega_2 = 372.7804$ rad/s in the simulation. This results in theoretical phase velocities of $c_1 = 29.1235$ and $c_2 = 23.2988$ m/s. The theoretical group velocity between the two traveling waves, according to $c_g = (\omega_2 - \omega_1)/(\gamma_2 - \gamma_1)$, is $c_g = 52.4222$ m/s.

The simulation was conducted from $t = \Delta t$ to $t = T = 0.2996$ sec with a sampling time of $\Delta t = 5.3936e-4$ sec (or a sampling rate of about 1.85 kHz). There were $M = 200$ spatial samples spaced by $\Delta x = 1.57$ cm in the medium, spanning $x = 0$ to $x = L = 3.1416$ m. Six-bit noise was added to the displacement ensemble. This noise was uniformly distributed in the interval $[-\epsilon, \epsilon]$, where $\epsilon = 2^{-6}$ times the maximum recorded beam displacement, and was generated by the Matlab command, 'rand'.

COD was applied, and the modal coordinates were isolated. The modal frequencies were estimated by fast Fourier transform (FFT) and from whirl rates, in the complex plane, of the complex modal coordinates and the complex modal vectors. The modal frequency estimates by whirl rate were $\omega_{1w} = 584.9966$ and $\omega_{2e} = 375.5177$ radians per second, and the estimated wave numbers were $\gamma_{1w} = 20.0000$ and $\gamma_{2w} = 16.0021$ radians per meter. The resulting phase velocity estimates were $c_{1e} = 29.2498$ and $c_{2e} = 23.4668$ m/s, and the group velocity estimate was $c_{ge} = 52.3803$ m/s.

2. Numerical Example: A Disturbed Infinite Euler-Bernoulli beam

The simulated steel beam had the same cross-section parameters as in the above example. The measurement interval was $L = 1.6$ m, and the spacing of the measurements was 1 cm. As such, there were 160 virtual sensors. With this measurement geometry, sampling principles imply that the minimum and maximum detectable wave numbers are $\gamma_{min} = 3.9270$ rad/m and $\gamma_{max} = 100\pi = 314.15$ rad/m. Based on the relationship between phase velocity, wave number, and frequency in an Euler-Bernoulli beam of the given geometry, the minimum and maximum detectable wave frequencies are $\omega_{min} = 22.4560$ rad/s and $\omega_{max} = 1.4372e + 05$ rad/s (or 22.874 kHz).

The simulation took place at a sampling interval of $\Delta t = 3.5437e - 05$ seconds (or 28.216 kHz) for a time record duration of $T = 0.2903$ seconds, and the displacements were recorded at uniform a spatial interval of $\Delta x = 1$ cm. (For real experiments in our lab, our sampling frequency will probably be lower, and therefore be the limit on detectable wave frequency and wave number.) The initial conditions were given as a Gaussian distribution such that $y(x, 0) = f_0 e^{-x^2/4b_0^2}$, where parameters $b = 1/100$ m$^{1/2}$ and $f_0 = 1$ mm. The initial velocities were zero. As such, the response of the beam is [2].

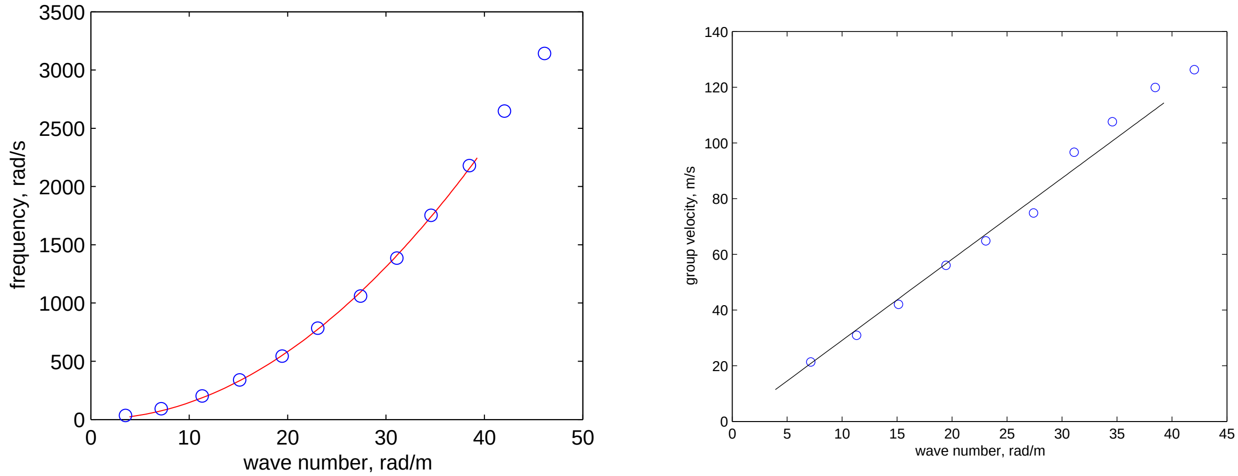
$$y(x,t) = 4^{1/4} f_0 b s^{1/4}(t) e^{-x^2 b^2 s(t)} \cos\left(atx^2 s(t) - \phi(t)\right) \tag{1}$$

where

$$s(t) = \frac{1}{4(b^4 + a^2t^2)}, \qquad \phi(t) = -\tan^{-1}\frac{at}{b^2}. \tag{2}$$

Random noise was added to each ensemble value. The noise was uniformly distributed over the interval $(-\epsilon, \epsilon)$, where $\epsilon = 2^{-6}$ times the largest value in the ensemble. Since the largest value in the ensemble is considerably larger than most ensemble values, as a lot of wave energy propagated off of the interval during the simulation, the noise level is quite significant. The mean of each sampled point history was removed. COD was applied to the complex ensemble, and the complex modes and complex modal coordinates were generated.

The spatial whirl rates of the complex modes were used to estimate wave numbers, and the temporal whirl rates of the complex modal coordinates, in the time range for which the oscillation persisted, were used to estimate frequencies. The estimated frequencies are



(a)

(b)

Figure 1: (a) The frequency versus wave number for the extracted modes (∘ symbols) compared to the theoretical curve (red line). (b) Group velocity versus wave number for the extracted modes (∘ symbols) compared to the theoretical curve (black line).

plotted against the estimated wave numbers in the circle symbols in Figure 1(a) along with the theoretical curve for the Euler-Bernoulli beam. These results are very similar to those of the noise free case (not shown). Also, the group velocities were computed using a finite difference approximation to the theory $c_g = d\omega/d\gamma$ [2], and are shown (circles) versus wave number in Figure 1(b) with comparison to Euler-Bernoulli beam theory (solid line).

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4. References

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